

One-loop Euclidean Einstein-Weyl gravity in de Sitter universe

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Abstract

By making use of the background field method, the one-loop quantization for Euclidean Einstein-Weyl quadratic gravity model on the de Sitter universe is investigated. Using generalized zeta function regularization, the on-shell and off-shell one-loop effective actions are explicitly obtained and one-loop renormalizability, as well as the corresponding one-loop renormalization group equations, are discussed. The so called critical gravity is also considered.

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1 Introduction

Recent astrophysical data indicate that our universe is currently in a phase of accelerated expansion. The physical origin of this acceleration is not completely understood and the related issue is commonly called the dark energy problem.

Several possible explanations have been proposed in the literature. One of them is based on the use of modified gravitational models, the simplest one consisting in the inclusion of a small and positive cosmological constant. Such model works quite well, according to the most recent data, but nevertheless it has some drawbacks (see, for example [1, 2, 3, 4] and reference therein).

Roughly speaking, the idea underlying modified gravity models is that the Einstein-Hilbert action gives only an approximate low energy contribution to gravitation and additional terms depending on the quadratic curvature invariants should necessarily be included. This idea is quite old since it was already contained in the seminal paper [5], where quadratic terms in the curvature, justified by quantum effects, were added to the Einstein-Hilbert Lagrangian (for a review, see [6]). The important recent finding is that the inclusion of suitable higher order contributions may realize not only the actual accelerated expansion, but also the early time inflation epoch [7]. Within this context, the de Sitter (dS) space-time plays a fundamental role, being able to provide an acceleration at different stages in the cosmological set up.

In previous papers [8, 9, 10, 11], $f(R)$ gravity models and a non local Gauss-Bonnet gravity model at one-loop level in a de Sitter background have been investigated. A similar program for the case of pure Einstein gravity was initiated in Refs. [12, 13, 14] (see also [15, 16]). Furthermore, such approach also suggests a possible way of understanding the cosmological constant issue [14]. Hence, the study of one-loop generalized modified gravity is a natural step to be undertaken for the completion of such program, with the aim to better understand the role and the origin of quadratic corrections in the curvature. An alternative approach, which is in some sense alternative, has been proposed by Rueter and collaborators [17], see also the review paper [18], and [19], in which quantum gravity effects in astrophysics and cosmology are presented.

In the present paper, we will investigate in some detail a model described by a Lagrangian density which depends on geometric quadratic invariants. The quantization of quadratic models of gravity has been discussed in many papers, and in particular studied in detail on flat space in the seminal paper [20]. A preliminary discussion of a quadratic model based on one-loop on-shell results has been presented in [21].

Here we start with the classical Euclidean gravitational action

$$I_E[g] = - \int d^4x \sqrt{g} F(R, P, Q) = - \int d^4x \sqrt{g} [M^2 R - 2\alpha + b(R^2 - 3P) + \beta G], \quad (1.1)$$

where b and β are dimensionless parameters, M^2 a mass-squared parameter playing the role of gravitational coupling constant, and α a “cosmological constant” dimensional term. By G we indicate the Gauss-Bonnet topological invariant which, in 4-dimensions, does not contribute to the classical field equations. For this reason, the action in (1.1) is classically equivalent to the so called Einstein-Weyl gravity, since the quadratic Weyl invariant W and the Gauss-Bonnet invariant G are related by

$$G - W = \frac{2}{3}(R^2 - 3P), \quad G = R^2 - 4P + Q, \quad W = \frac{1}{3}R^2 - 2P + Q, \quad (1.2)$$

P and Q being

$$P = R^{ij}R_{ij}, \quad Q = R^{ijrs}R_{ijrs}, \quad i, j, r, s = 0, 1, 2, 3. \quad (1.3)$$

As we already said above, at classical level the Gauss-Bonnet term does not play any role and could be dropped off but, as we shall see in the following, it will play an important role at quantum level.

The action in the form (1.1) is quite useful in order to discuss the so called “critical gravity”, which corresponds to a particular choice of the b parameter. An extensive study of 4-dimensional

critical gravity, in the presence of a negative cosmological constant, has been recently presented in Refs. [22, 23, 24, 25], where additional relevant references can be found.

It should be noted that one-loop Euclidean quantum gravity in a de Sitter background—as was fully exploited in [14] for the case of Einstein’s gravity in the presence of a cosmological term—presents some peculiar aspects within the background field method. First, working with the Euclidean version $S(4)$, one is dealing with a geometric background associated with a compact manifold without a boundary. This means that the volume is finite and can be expressed as a function of the constant Ricci curvature, which may be chosen as background field. A second important remark is that, in order to discuss the one-loop renormalizability of the model, as well as the related renormalization group equations, one is forced to work with the off-shell one-loop effective action. As a consequence, the Landau gauge appears to be the most convenient one. Besides, the usual effective action calculated in this gauge coincides with the Vilkovisky-De Witt effective action (see, for example, [6]).

Such approach should be compared with the more traditional one, nicely reviewed in [26], where the Sakharov induced gravity approach [27] and its modern variants [28] have been discussed too. Conformal gravity has been discussed in [29]. The ghost absence issue for a very general gravitational quadratic model on Minkowski space-time has been recently investigated in [30]. Furthermore, alternative approach is presented in [31].

Regarding to the choice of regularization, since we are dealing with non flat space-time, it is almost mandatory (or at least very convenient) to make use of a variant of the generalized zeta-function regularization [32, 33] (see also [34, 35, 36]), and the associated heat-kernel techniques [37, 38, 39]. In this way, one may evaluate the one-loop effective action and then study the possibility of stabilization of the de Sitter background by quantum effects.

The paper is organized as follows. Section II contains the evaluation of the quantum fluctuation operators relevant for the one-loop calculations to be carried out. In Section III, the off-shell one loop partition function is presented and the corresponding one loop renormalization is discussed. Finally, Section IV is devoted to conclusions.

2 Quantum field fluctuations around maximally symmetric instantons

In this Section we will discuss the one-loop quantization of the model in (1.1) on a maximally symmetric space (see, for instance [6]). To start with, we consider the Euclidean gravitational action in (1.1) and, for convenience, we separate linear and quadratic terms

$$F(R, P, Q) = f(R) + b(R^2 - 3P) + \beta G, \quad f(R) = M^2 R - 2\alpha. \quad (2.1)$$

The model admits a constant Ricci curvature solution R_0 . In fact, the general equation for the existence of de Sitter solution [40, 41]

$$\left[\left(\frac{1}{2} R \frac{\partial}{\partial R} + P \frac{\partial}{\partial P} + Q \frac{\partial}{\partial Q} - 1 \right) F(R, P, Q) \right]_{R=R_0} = 0, \quad (2.2)$$

is trivially satisfied, and reads

$$f(R_0) - \frac{1}{2} R_0 f'(R_0) = 0 \quad \implies \quad R_0 = \frac{4\alpha}{M^2}. \quad (2.3)$$

We are interested in studying quantum fluctuations around the Euclidean dS instanton S^4 with positive constant scalar curvature R_0 . This is a maximally symmetric space having covariant conserved curvature tensors. Its metric may be written in the form

$$ds_E^2 = d\tau^2 (1 - H_0^2 r^2) + \frac{dr^2}{(1 - H_0^2 r^2)} + r^2 dS_2^2, \quad (2.4)$$

dS^2 being the metric of the two-dimensional sphere S^2 and H_0 the Hubble constant. The finite volume is given by

$$V(S^4) = \frac{384\pi^2}{R_0^2}, \quad R_0 = 12H_0^2, \quad G_0 = 24H_0^4, \quad (2.5)$$

while the Riemann and Ricci tensors are

$$R_{ijrs}^{(0)} = \frac{R_0}{12} (g_{ir}^{(0)} g_{js}^{(0)} - g_{is}^{(0)} g_{jr}^{(0)}) , \quad R_{ij}^{(0)} = \frac{R_0}{4} g_{ij}^{(0)} . \quad (2.6)$$

Now let us consider small fluctuations around the maximally symmetric instanton. For the sake of completeness, we consider the general action discussed in [21], but linear in P, Q . Then, we shall restrict to the action (1.1), at the end of the computation. For simplicity, we also put $M^2 = 1$. When necessary the right units will be easily recovered by dimensional analysis.

We set

$$g_{ij} \longrightarrow g_{ij} + h_{ij} , \quad g^{ij} \longrightarrow g^{ij} - h^{ij} + h^{ik} h_k^j + \mathcal{O}(h^3) , \quad h = g^{ij} h_{ij} , \quad (2.7)$$

where from now on $g_{ij} \equiv g_{ij}^{(0)}$ is the metric of the maximally symmetric space and, as usual, indices are lowered and raised by means of such metric.

Up to second order in h_{ij} , one has

$$\sqrt{g} \longrightarrow \sqrt{g} \left[1 + \frac{1}{2} h + \frac{1}{8} h^2 - \frac{1}{4} h_{ij} h^{ij} + \mathcal{O}(h^3) \right] \quad (2.8)$$

and

$$\begin{aligned} R \sim R_0 - \frac{R_0}{4} h + \nabla_i \nabla_j h^{ij} - \Delta h \\ + \frac{R_0}{4} h^{jk} h_{jk} - \frac{1}{4} \nabla_i h \nabla^i h - \frac{1}{4} \nabla_k h_{ij} \nabla^k h^{ij} + \nabla_i h_k^i \nabla_j h^{jk} - \frac{1}{2} \nabla_j h_{ik} \nabla^i h^{jk} , \end{aligned} \quad (2.9)$$

where ∇_k represents the covariant derivative in the unperturbed metric g_{ij} . More complicated expressions are obtained for the other invariants P, Q , but for our aims it is not necessary to write them explicitly.

By performing a Taylor expansion of the Lagrangian around de Sitter metric, up to second order in h_{ij} , we get

$$I_E[g] \sim - \int d^4x \sqrt{g} \left[F(R_0, P_0, Q_0) + \frac{hX}{2} + \mathcal{L}_2 \right] , \quad (2.10)$$

where \mathcal{L}_2 represents the second-order contribution and X vanishes when the de Sitter existence condition (2.2) is satisfied. For our particular model, $X = [f(R_0) - (1/2)R_0 f'(R_0)]/M^2$.

It is convenient to carry out the standard expansion of the tensor field h_{ij} in irreducible components [14], namely

$$h_{ij} = \hat{h}_{ij} + \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \nabla_j \sigma + \frac{1}{4} g_{ij} (h - \Delta \sigma) , \quad (2.11)$$

where σ is the scalar component, while ξ_i and \hat{h}_{ij} are the vector and tensor components, with the following properties

$$\nabla_i \xi^i = 0 , \quad \nabla_i \hat{h}^{ij} = 0 , \quad \hat{h}_i^i = 0 . \quad (2.12)$$

In terms of the irreducible components of the h_{ij} field, the Lagrangian density, disregarding total derivatives, becomes

$$\mathcal{L}_2 = \mathcal{L}_{hh} + 2\mathcal{L}_{h\sigma} + \mathcal{L}_{\sigma\sigma} + \mathcal{L}_V + \mathcal{L}_T , \quad (2.13)$$

where $\mathcal{L}_{hh}, \mathcal{L}_{h\sigma}, \mathcal{L}_{\sigma\sigma}$ represent the scalar contribution (a 2×2 matrix), while \mathcal{L}_V and \mathcal{L}_T represent the vector and tensor contributions, respectively. One has

$$\begin{aligned} \mathcal{L}_{hh} = h & \left[\frac{1}{32} F_{RR} R_0^2 - \frac{1}{32} F_R R_0 + \frac{X}{16} - \frac{3}{32} F_R \Delta + \frac{1}{16} F_P R_0 \Delta + \frac{1}{16} F_Q R_0 \Delta \right. \\ & \left. + \frac{3}{16} F_{RR} R_0 \Delta + \frac{3}{16} F_P \Delta^2 + \frac{3}{16} F_Q \Delta^2 + \frac{9}{32} F_{RR} \Delta^2 \right] h, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \mathcal{L}_{h\sigma} = h & \left[-\frac{1}{16} F_{RR} R_0^2 \Delta + \frac{1}{16} F_R R_0 \Delta - \frac{1}{8} F_P R_0 \Delta^2 \right. \\ & - \frac{1}{8} F_Q R_0 \Delta^2 - \frac{3}{8} F_{RR} R_0 \Delta^2 + \frac{3}{16} F_R \Delta^2 \\ & \left. - \frac{3}{8} F_P \Delta^3 - \frac{3}{8} F_Q \Delta^3 - \frac{9}{16} F_{RR} \Delta^3 \right] \sigma, \end{aligned} \quad (2.15)$$

$$\begin{aligned} \mathcal{L}_{\sigma\sigma} = \sigma & \left[\frac{1}{32} F_{RR} R_0^2 \Delta^2 - \frac{1}{16} X R_0 \Delta - \frac{1}{32} F_R R_0 \Delta^2 - \frac{3}{16} X \Delta^2 \right. \\ & + \frac{1}{16} F_P R_0 \Delta^3 + \frac{1}{16} F_Q R_0 \Delta^3 + \frac{3}{16} F_{RR} R_0 \Delta^3 - \frac{3}{32} F_R \Delta^3 \\ & \left. + \frac{3}{16} F_P \Delta^4 + \frac{3}{16} F_Q \Delta^4 + \frac{9}{32} F_{RR} \Delta^4 \right] \sigma, \end{aligned} \quad (2.16)$$

$$\mathcal{L}_V = \xi^k \left[\frac{1}{8} R_0 X + \frac{1}{2} X \Delta \right] \xi_k, \quad (2.17)$$

$$\begin{aligned} \mathcal{L}_T = h^{ij} & \left[-\frac{1}{72} F_P R_0^2 + \frac{1}{36} F_Q R_0^2 - \frac{1}{24} F_R R_0 - \frac{1}{4} X + \frac{1}{4} F_R \Delta \right. \\ & \left. + \frac{1}{24} F_P R_0 \Delta - \frac{1}{3} F_Q R_0 \Delta + \frac{1}{4} F_P \Delta + F_Q \Delta^2 \right] \hat{h}_{ij}. \end{aligned} \quad (2.18)$$

where $\Delta = g^{ij} \nabla_i \nabla_j$ is the Laplace-Beltrami operator in the unperturbed metric g_{ij} , which is a solution of the field equations, but only if $X = 0$. We have written the above expansions around a maximally symmetric space, which in principle would not be a solution. This means, in other words, that the function $f(R)$ can be arbitrary. In the latter expression F_R, F_{RR} represent the first and second derivatives of $F(R, P, Q)$ with respect to R evaluated on de Sitter metric g_{ij} . And similarly for F_P, F_Q .

As is well known, invariance under diffeomorphisms renders the operator in the (h, σ) sector not invertible. One needs a gauge fixing term and a corresponding ghost compensating term. Here we choose the harmonic gauge, that is

$$\chi_j = -\nabla_i h_j^i - \frac{1}{2} \nabla_j h = 0, \quad (2.19)$$

and the gauge fixing term

$$\mathcal{L}_{gf} = \frac{1}{2} \chi^i G_{ij} \chi^j, \quad G_{ij} = \gamma g_{ij}. \quad (2.20)$$

The corresponding ghost Lagrangian reads [6]

$$\mathcal{L}_{gh} = B^i G_{ik} \frac{\delta \chi^k}{\delta \varepsilon^j} C^j, \quad (2.21)$$

where C_k and B_k are the ghost and anti-ghost vector fields respectively, while $\delta \chi^k$ is the variation of the gauge condition due to an infinitesimal gauge transformation of the field. In this case, it reads

$$\delta h_{ij} = \nabla_i \varepsilon_j + \nabla_j \varepsilon_i \quad \Longrightarrow \quad \frac{\delta \chi^i}{\delta \varepsilon^j} = g_{ij} \Delta + R_{ij}. \quad (2.22)$$

Neglecting total derivatives, one has

$$\mathcal{L}_{gh} = B^k \gamma \left(\Delta + \frac{R_0}{4} \right) C_k. \quad (2.23)$$

In irreducible components one finally obtains

$$\begin{aligned} \mathcal{L}_{gf} = & \frac{\gamma}{2} \left[\xi^k \left(\Delta + \frac{R_0}{4} \right)^2 \xi_k + \frac{3\rho}{8} h \left(\Delta + \frac{R_0}{3} \right) \Delta \sigma \right. \\ & \left. - \frac{\rho^2}{16} h \Delta h - \frac{9}{16} \sigma \left(\Delta + \frac{R_0}{3} \right)^2 \Delta \sigma \right] \end{aligned} \quad (2.24)$$

$$\mathcal{L}_{gh} = \gamma \left[\hat{B}^k \left(\Delta + \frac{R_0}{4} \right) \hat{C}_k + \frac{\rho-3}{2} \hat{b} \left(\Delta - \frac{R_0}{\rho-3} \right) \Delta \hat{c} \right], \quad (2.25)$$

where ghost irreducible components are defined by

$$\begin{aligned} C_k &= \hat{C}_k + \nabla_k \hat{c}, & \nabla_k \hat{C}^k &= 0, \\ B_k &= \hat{B}_k + \nabla_k \hat{b}, & \nabla_k \hat{B}^k &= 0. \end{aligned} \quad (2.26)$$

3 Off-shell one-loop effective action

In order to compute the one-loop contributions to the effective action one has to consider the path integral for the bilinear part, $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{gf} + \mathcal{L}_{gh}$, of the total Lagrangian and take into account the Jacobian due to the change of variables with respect to the original ones. In this way, one gets [14, 6]

$$\begin{aligned} Z^{(1)} &= (\det G_{ij})^{-1/2} \int D[h_{ij}] D[C_k] D[B^k] \exp \left(- \int d^4x \sqrt{g} \mathcal{L} \right) \\ &= (\det G_{ij})^{-1/2} \det J_1^{-1} \det J_2^{1/2} \\ &\quad \times \int D[h] D[\hat{h}_{ij}] D[\xi^j] D[\sigma] D[\hat{C}_k] D[\hat{B}^k] D[c] D[b] \exp \left(- \int d^4x \sqrt{g} \mathcal{L} \right), \end{aligned} \quad (3.1)$$

where J_1 and J_2 are the Jacobians coming from the change of variables in the ghost and tensor sectors, respectively [14]. They read

$$J_1 = \Delta_0, \quad J_2 = \left(\Delta_1 + \frac{R_0}{4} \right) \left(\Delta_0 + \frac{R_0}{3} \right) \Delta_0, \quad (3.2)$$

and the determinant of the operator G_{ij} is trivial in this case. Here and in the following $\Delta_0, \Delta_1, \Delta_2$, represent the Laplacian acting on scalars, vectors and tensors, respectively.

Due to the presence of curvature, the Euclidean gravitational action is not bounded from below, because arbitrary negative contributions can be induced on R by conformal rescaling of the metric. For this reason we have also used the Hawking prescription of integrating over imaginary scalar fields. Furthermore, the problem of the presence of additional zero modes introduced by the decomposition (2.11) can be treated by making use of the method presented in Ref. [14].

Now, for the action (1.1) a straightforward computation leads to the off-shell one-loop contribution to the “partition function”. In the Landau gauge, $\rho = 1, \gamma \rightarrow \infty$, with $X = R_0/2 - 2\alpha/M^2$, we get

$$\Gamma_{off-shell} = I_E(g) + \Gamma_{off-shell}^{(1)}, \quad I_E(g) = 96\pi^2 \left(\frac{2M^2}{R_0} + b \right) + 64\pi^2 \beta, \quad (3.3)$$

$$\begin{aligned}
\Gamma_{off-shell}^{(1)} &= \sum_i \frac{1}{2} \log \det \frac{L_i}{\mu^2} = \frac{1}{2} \log \det \left(\frac{1}{\mu^2} \left[-\Delta_0 - \frac{2\alpha}{M^2} \right] \right) \\
&\quad - \frac{1}{2} \log \det \left(\frac{1}{\mu^2} \left[-\Delta_1 - \frac{R_0}{4} \right] \right) - \frac{1}{2} \log \det \left(\frac{1}{\mu^2} \left[-\Delta_0 - \frac{R_0}{2} \right] \right) \\
&\quad + \frac{1}{2} \log \det \left(\frac{1}{\mu^2} [-\Delta_2 - Y_+] \right) + \frac{1}{2} \log \det \left(\frac{1}{\mu^2} [-\Delta_2 - Y_-] \right), \tag{3.4}
\end{aligned}$$

where

$$Y_{\pm} = \frac{1}{12} \left(-3R_0 - \frac{2M^2}{b} \pm \frac{1}{b} \sqrt{96b\alpha + 4M^4 - 20bM^2R_0 + b^2R_0^2} \right). \tag{3.5}$$

As usual, an arbitrary renormalization parameter $1/\mu^2$ has been introduced for dimensional reasons. As expected, the parameter β does not appear in the latter expression, since the Gauss-Bonnet invariant does not give contributions to the field equations, but it gives a constant contribution to the classical action, which will actually play an important role in the renormalization procedure.

3.1 On-shell one-loop effective action

As is well known, the on-shell effective action does not have to depend on the gauge and, in fact, setting $X = 0$, that is $M^2R_0 - 4\alpha = 0$, we get

$$\begin{aligned}
\Gamma_{on-shell} &= 96\pi^2 \left(\frac{2M^2}{R_0} + b \right) + 64\pi^2\beta - \frac{1}{2} \log \det \left(\frac{1}{\mu^2} \left[-\Delta_1 - \frac{R_0}{4} \right] \right) \\
&\quad + \frac{1}{2} \log \det \left(\frac{1}{\mu^2} \left[-\Delta_2 + \frac{R_0}{6} \right] \right) + \frac{1}{2} \log \det \left(\frac{1}{\mu^2} \left[-\Delta_2 + \frac{R_0}{3} + \frac{M^2}{3b} \right] \right). \tag{3.6}
\end{aligned}$$

The above expression is only formal, and one needs regularization. For the moment, let us imagine to be dealing with the finite part of such an effective action.

We observe that there exists a “critical” value for b for which all spin excitations become “massless”. In fact, choosing

$$b = b_{crit} = -\frac{2M^2}{R_0} = -\frac{M^4}{2\alpha}, \tag{3.7}$$

the effective action simplifies to

$$\Gamma_{crit} = 64\pi^2\beta - \frac{1}{2} \log \det \left(\frac{1}{\mu^2} \left[-\Delta_1 - \frac{R_0}{4} \right] \right) + \log \det \left(\frac{1}{\mu^2} \left[-\Delta_2 + \frac{R_0}{6} \right] \right). \tag{3.8}$$

In contrast to the AdS case, in Euclidean dS space $R_0 > 0$, and so $b_{crit} < 0$.

The stability of the dS solution can be investigated by looking at the spectra of the Laplace-type operators and it then follows that all eigenvalues are non-negative as in general relativity, with the possible presence of a zero mode [14, 9]. As a consequence, dS background space is stable, in agreement with the classical analysis presented in [40, 41].

But what about the stability of the critical values with respect to renormalization? According to the background field method one should work at the off-shell level. Nevertheless, one may try an on-shell, one-loop renormalization, by observing that β might become a “bare” constant, and its redefinition may contain all counterterms necessary in order to cancel the on-shell one-loop divergences coming from the functional determinants. In general, using a variant of the zeta function regularization procedure [21], at one-loop level one has

$$\Gamma(\mu, \varepsilon) = I_E(\mu, \varepsilon) - \frac{1}{2} \sum_i \left[\frac{\zeta(0|L_i)}{\varepsilon} + \zeta(0|L_i) \log \mu^2 + \zeta'(0|L_i) \right], \tag{3.9}$$

where the summation is over all Laplace-type operators appearing in the one-loop contribution to the action.

For the critical gravity in (3.8), we have $L_1 = -\Delta_1 - R_0/4$ and $L_2 = -\Delta_2 + R_0/6$, thus

$$\begin{aligned}\Gamma(\mu, \varepsilon) &= 64\pi^2 \left(\frac{\beta_0}{\varepsilon} + \beta(\mu) \right) + \frac{1}{2\varepsilon} [\zeta(0|L_1) - 2\zeta(0|L_2)] \\ &\quad + \log \mu [\zeta(0|L_1) - 2\zeta(0|L_2)] + \frac{1}{2} \zeta'(0|L_1) - \zeta'(0|L_2),\end{aligned}\quad (3.10)$$

where β_0 is the bare coupling constant and $\beta(\mu)$ the running one. Making a suitable choice for β_0 , one has the renormalized on-shell effective critical action

$$\Gamma_{crit}(\mu) = 64\pi^2 \beta(\mu) + \log \mu [\zeta(0|L_1) - 2\zeta(0|L_2)] + \frac{1}{2} \zeta'(0|L_1) - \zeta'(0|L_2). \quad (3.11)$$

The eigenvalues of the Laplace type operators on $SO(4)$ are well known and in this way it is possible to compute the zeta-functions appearing in the expression above explicitly. In particular, $\zeta(0|L_1) = -191/30$ and $\zeta(0|L_2) = 89/9$, while $\zeta'(0|L_1)$ and $\zeta'(0|L_2)$ are computable expressions independent on μ . The usual imposition

$$\mu \frac{d\Gamma(\mu)}{d\mu} = 0, \quad (3.12)$$

gives rise to the renormalization group equation for critical gravity, in the form

$$\mu \frac{d\beta(\mu)}{d\mu} = 2\zeta(0|L_2) - \zeta(0|L_1) \sim 26 > 0. \quad (3.13)$$

This is the only running coupling constant and, thus, on-shell critical gravity seems to be stable at the one-loop level. But this is not really conclusive since, strictly, the issue of criticality depends on the on-shell expression.

3.2 Off-shell one-loop renormalization

As far as the off-shell one-loop renormalization is concerned, the situation is completely different with respect to the previous one and apparently there is no room for the notion of criticality.

Again, the starting point is the equation in (3.9), but now the classical action contains all the bare quantities, which generate the counterterms for absorbing the one-loop divergencies. It reads

$$\begin{aligned}I_E(\mu, \varepsilon) &= 384\pi^2 \left[\frac{M^2(\mu, \varepsilon)}{R_0} - \frac{2\alpha(\mu, \varepsilon)}{R_0^2} + \frac{b(\mu, \varepsilon)}{4} + \frac{\beta(\mu, \varepsilon)}{6} \right] \\ &= 384\pi^2 \left[\frac{M^2(\mu)}{R_0} - \frac{2\alpha(\mu)}{R_0^2} + \frac{b(\mu)}{4} + \frac{\beta(\mu)}{6} \right] + \frac{1}{\varepsilon} \left[\frac{A_1}{R_0} + \frac{B_1}{R_0^2} + C_1 \right],\end{aligned}\quad (3.14)$$

where we have separated the finite and divergent parts of the coupling constants by means of suitable finite quantities A_1, B_1, C_1 independent of R_0 . On the other hand, a direct computation shows that

$$\sum_i \zeta(0|L_i) = \left[\frac{A(\mu)}{R_0} + \frac{B(\mu)}{R_0^2} + C(\mu) \right], \quad (3.15)$$

where L_i are all Laplace-type operators in (3.4) and $A(\mu), B(\mu), C(\mu)$ are finite functions depending on the renormalized running coupling constants $M^2(\mu), \alpha(\mu), b(\mu)$. They read

$$A(\mu) = \frac{8\alpha(\mu)}{M^2(\mu)}, \quad B(\mu) = \frac{20M^4(\mu)}{3b^2(\mu)} + \frac{80\alpha(\mu)}{b(\mu)} + \frac{48\alpha^2(\mu)}{M^4(\mu)}, \quad C(\mu) = \frac{1763}{90}. \quad (3.16)$$

The model is one-loop renormalizable since all one-loop divergences can actually be absorbed by an appropriate choice of A_1, B_1, C_1 . The finite, renormalized one-loop effective action reads

$$\Gamma(\mu) = 384\pi^2 \left[\frac{M^2(\mu)}{R_0} - \frac{2\alpha(\mu)}{R_0^2} + \frac{b(\mu)}{4} \right] + \log \mu \left[\frac{A(\mu)}{R_0} + \frac{B(\mu)}{R_0^2} + C(\mu) \right] + Z, \quad (3.17)$$

where we have dropped the parameter $\beta(\mu)$ because here it does not play any role, and we have set $Z = -\frac{1}{2} \sum_i \zeta'(0|L_i)$. This is the finite part of the functional determinant which does not depend on μ and in principle can be explicitly evaluated.

As above, the one-loop renormalization group equations can be obtained by means of (3.12). To this aim it is convenient to introduce the dimensionless variable $\rho = \log \frac{\mu}{\mu_0}$, μ_0 being a reference low energy scale. From (3.12) we obtain the three differential equations

$$\begin{cases} \frac{db}{d\rho} = c_b, \\ \frac{dM^2}{d\rho} = \frac{\alpha}{48\pi^2 M^2}, \\ \frac{d\alpha}{d\rho} = -\frac{1}{192\pi^2 b^2} \left(12 \frac{\alpha^2 b^2}{M^4} + 20\alpha b + \frac{5}{3} M^4 \right), \end{cases} \quad c_b = \frac{1763}{8640\pi^2}, \quad (3.18)$$

where all coupling constants are functions of ρ . Solving the system of differential equations above, we finally get

$$\begin{cases} b(\rho) = c_b \rho + c_0, \\ M^2(\rho) = c_1 (\rho + c_0)^{p_1 - p_2} [(\rho + c_0)^{10p_2} + c_2]^{1/5}, \\ \alpha(\rho) = 48\pi^2 \frac{M^4(\rho)}{(\rho + c_0)} \left[p_1 - p_2 + \frac{2p_2}{1 + c_2(\rho + c_0)^{-10p_2}} \right], \end{cases} \quad \begin{cases} p_1 = \frac{863}{17630}, \\ p_2 = \frac{\sqrt{474769}}{17630}. \end{cases} \quad (3.19)$$

The integration constants c_0, c_1, c_2 depend on the initial conditions and we assume all of them to be non negative. Moreover, to simplify the discussion, from now on we shall take $c_2 = 0$. With this assumption we get

$$M^2(\rho) = c_1 (\rho + c_0)^p, \quad \alpha(\rho) = 48\pi^2 p \frac{M^4(\rho)}{\rho + c_0}, \quad p = p_1 + p_2 \sim 0.09, \quad (3.20)$$

and the one-loop, running, gravitational coupling constant reads

$$G(\rho) = \frac{1}{16\pi M^2(\rho)} = \frac{1}{16\pi c_1 (\rho + c_0)^p}, \quad (3.21)$$

while the one-loop, running, cosmological constant is

$$\Lambda(\rho) = \frac{\alpha(\rho)}{M^2(\rho)} = 48\pi^2 p c_1 (\rho + c_0)^{p-1}. \quad (3.22)$$

As a result, there is no Landau pole, and at large energy $\mu \gg \mu_0$ or equivalently $\rho \rightarrow \infty$, one has that both $G(\rho)$ and $\Lambda(\rho)$ go to zero. This property is the analogue on de Sitter space of the well known gravitational asymptotic freedom for quadratic gravity [42, 43, 44, 45, 46].

Furthermore, we may assume general relativity to be valid at low energy, that is $\mu \sim \mu_0$ or equivalently $\rho \sim 0$, then

$$G(\rho)|_{\rho \rightarrow 0} = G_N \quad \implies \quad c_1 = \frac{c_0^{-p}}{16\pi G_N}, \quad M^2(\rho) = \frac{1}{16\pi G_N} \left(1 + \frac{\rho}{c_0} \right)^p, \quad (3.23)$$

$G_N = G(0)$ being the Newton constant.

To conclude this section, we write down the effective field equation given by

$$\frac{\partial \Gamma}{\partial R_0} = 0. \quad (3.24)$$

The solution can be written in the implicit form

$$R_0 = \frac{1}{1 - \frac{A\rho}{384\pi^2 M^2}} \left[\frac{4\alpha}{M^2} - \frac{B\rho}{192\pi^2 M^2} + \frac{R_0^3}{384\pi^2 M^2} \frac{\partial Z}{\partial R_0} \right]. \quad (3.25)$$

This is a quite complicated expression in the unknown variable R_0 . Of course, at low energy $\rho \sim 0$, $R_0 \ll M^2$, one gets the classical solution

$$R_0 \sim \frac{4\alpha(0)}{M_P^2(0)} = 4\Lambda(0), \quad (3.26)$$

but in principle other regimes can be studied.

4 Conclusions

In this paper, the Einstein gravity plus a quadratic gravitational Weyl term has been investigated by computing the corresponding one-loop quantum corrections by means of the background field method. As a classical background we have considered the de Sitter one for its potentially very important physical applications. The one-loop calculation has been performed in the Euclidean sector, where the classical background is the compact manifold S^4 . In the calculation, due to the fact that we are working with a non flat background, we are forced to make use of (a variant of) zeta-function regularization. In the presence of this compact curved manifold, the one-loop effective action, and also the ensuing one-loop renormalization group equations, have been computed and carefully investigated. On shell, the associated quadratic critical gravity has been discussed. In order to investigate the role of the one-loop corrections, we have to consider the off-shell one-loop effective action and so, to get rid of the gauge dependence, the Landau gauge has been used. In this way the critical conditions are lost, in general. In fact, also in the simplified case we have considered ($c_2 = 0$), the critical ratio $\frac{M^4(\rho)}{2\alpha(\rho)} \sim (\rho + c_0)$, which is not equal to $b(\rho)$. This means that the critical gravity condition are not stable under the one-loop renormalization flow. As a consequence, one might doubt about its relevance at least when the background is the compact manifold S^4 . In fact, in the anti de Sitter case (AdS) the situation could be completely different, because the Euclidean counterpart of AdS is the non compact hyperbolic manifold H^4 .

In conclusion, it should also be interesting to repeat this one-loop calculation in alternative extended gravity models, for example, in the so-called $f(T)$ gravity models, which depend on geometric invariants built up by using torsion field (for details see [47] and the references therein).

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